

Confidence Intervals and Upper Bounds for Small Signals in the Presence of Background Noise ¹

Wolfgang A. Rolke ^a

^a*Department of Mathematics, University of Puerto Rico - Mayagüez, Mayagüez,
PR 00681, USA,*

Postal Address: PO Box 5959, Mayagüez, PR 00681,

Tel: (787) 851-6212, Email: w_rolke@rumac.upr.clu.edu

Angel M. López ^b

^b*Department of Physics, University of Puerto Rico - Mayagüez, Mayagüez, PR
00681, USA*

Abstract

We discuss a new method for setting limits on small signals in the presence of background noise. The method is based on a combination of a two dimensional confidence region and the large sample approximation to the likelihood ratio test statistic. It automatically quotes upper limits for small signals and two-sided confidence intervals for larger samples. We show that this method gives the correct coverage and also has good power.

Key words: Maximum likelihood, profile likelihood, confidence regions, coverage, Monte Carlo, sensitivity

¹ This work was supported by the High Energy Physics Office (Grant DE-FG-97ER41045) and the EPSCoR Program (Grant DE-FG-94ER75764) of the US Department of Energy and the University of Puerto Rico.

1 Introduction

Finding confidence intervals or upper limits for small signals has recently attracted a great deal of attention. The paper by Feldman and Cousins [1] rekindled the interest in this area by providing a unified approach, that is, it based the choice of quoting an upper limit or a two-sided confidence interval on the data alone, without the experimenter having to make this decision. The unified approach uses the Neyman construction together with an ordering principle based on likelihood ratios. Subsequent papers such as Giunti [2] and Roe and Woodroffe [3] gave variations of this method, basing the ordering on other quantities. Common to all these methods is the need to have a fairly precise knowledge of the background rate, for example, from Monte Carlo simulations. Unfortunately, as we will see in section 4, these methods can fail when used in the presence of background uncertainty. A possible remedy is discussed in Cousins and Highland [4] where it is suggested to treat the background uncertainty as a systematic error. We will instead treat the background uncertainty as a statistical error and develop a method that is suitable in this situation. Our method is based on the likelihood ratio test statistic, together with an adjustment for those situations where there is very little (or no) signal observed. We will show that this method yields the correct coverage rate and that it has good power.

The background rate has to be estimated either from the data or through

Monte Carlo. Both of those methods have their strengths and their weaknesses. Using data requires choosing sidebands, and implicitly makes the assumption that the density generating the background events is the same in the signal region as it is in the sidebands. This leads to a quandary: If we choose a small sideband, this assumption seems more reasonable, but then we will also see fewer background events and therefore have a higher statistical uncertainty in the estimate of the background rate. Choosing a large region might yield higher statistics but makes the assumption of a linear background more tenuous. An alternative way to estimate the background rate is by Monte Carlo. One problem with this approach is that a good Monte Carlo is often difficult to do because we can never be completely sure that we have modeled all the relevant effects correctly. This is particularly true when searching for small signals since the efforts to reduce the background often mean one is probing the tails of distributions which may be difficult to model. Also, in High Energy Physics today running a complete Monte Carlo simulation of an experiment can be a formidable task from a computational point of view, and it might not be possible to run enough Monte Carlo to effectively eliminate the uncertainty in the background estimate. For these reasons, it is in many cases not possible to ignore the uncertainty in the background rate. Our method is well equipped to deal with the uncertainty that comes from limited statistics in both situations, those where the background is estimated from the data as well as those where the background is estimated using Monte Carlo.

In some cases there are two different sources of background. It turns out that our method can be extended fairly easily to this situation also, regardless of the method of estimation used.

2 A Description of the Method

In this section we will outline the basic ideas of this new method. We will need the following notation. Assume that we observe x events in a suitably chosen signal region and a total of y events in the background region. Here the background region can be chosen fairly freely and need not be contiguous. Furthermore, we assume that the ratio of the size of the background region to the size of the signal region is τ . For example, if we use two background regions of the same size as the signal region we get $\tau = 2$. Then a probability model for the data is given by

$$X \sim Pois(\mu + b), \quad Y \sim Pois(\tau b)$$

where μ is the signal rate, b is the background rate and $Pois$ is the usual Poisson distribution. We can assume X and Y to be independent and so

$$P_{\mu,b}(X = x, Y = y) = \frac{(\mu + b)^x}{x!} e^{-(\mu+b)} \cdot \frac{(\tau b)^y}{y!} e^{-\tau b}$$

2.1 A Confidence Region for μ and b

A common technique for constructing confidence regions (or intervals for one-dimensional parameters) is to find a corresponding hypothesis test and then to invert the test. We will start with a simultaneous hypothesis test for μ and b with the null hypothesis $H_0 : \mu = \mu_0, b = b_0$. The steps are as follows:

- (1) List all observations $(u_i, v_i), i = 1, \dots, K$, together with their probabilities, which are given by

$$P_{\mu_0, b_0}(X = u_i, Y = v_i) = \frac{(\mu_0 + b_0)^{u_i}}{u_i!} e^{-(\mu_0 + b_0)} \cdot \frac{(\tau b_0)^{v_i}}{v_i!} e^{-\tau b_0}$$

Here we use the values μ_0 and b_0 specified in the null hypothesis. We list only those observations which have a probability above a certain small threshold. In our algorithm we require the probabilities to be larger than 10^{-6} .

- (2) Sort all observations from the most likely to the most unlikely.
- (3) Find the partial sums from the largest to the k^{th} observation until you reach $1 - \alpha$, if the desired level of the test is α .
- (4) If the observed (x, y) appears in the list of possible observations before $1 - \alpha$ is reached, accept the null hypothesis, otherwise reject it.

This is in effect a Neyman construction like the one used in Feldman and Cousins [1], only we use the probabilities as the ordering quantities. The test simply checks whether or not the observation (x, y) is compatible with the

rates μ_0 and b_0 specified in the null hypothesis.

Using the likelihood ratios as the ordering quantity in a way similar to Feldman and Cousins [1] would in principle be superior, but unfortunately does not yield a viable method here because the list of "likely" observations is infinite.

The inversion of the hypothesis test then involves a search through all pairs (μ, b) . If a certain pair leads to the acceptance of the null hypothesis, we add it to the confidence region, otherwise we do not. As an example we have figure 1 where we show the confidence regions obtained for three observations. The boundaries of the confidence regions are somewhat ragged due to the discrete nature of the Poisson random variable.

2.2 A Confidence Interval for μ

Again we will start with a hypothesis test, but this time we will only fix the signal rate μ . The null hypothesis then becomes $H_0 : \mu = \mu_0$.

A popular test in Statistics for any kind of hypothesis test is the likelihood ratio test, which is based on the likelihood ratio test statistic Λ given in our problem by:

$$\Lambda(\mu_0; x, y) = \frac{\max \{l(\mu_0, b; x, y) : b \geq 0\}}{\max \{l(\mu, b; x, y) : \mu \geq 0, b \geq 0\}}$$

Here $l(\mu, b; x, y) = P_{\mu, b}(X = x, Y = y)$ is the likelihood function of μ and b

given the observation (x, y) . This test statistic can be thought of as the ratio of the best explanation for the data if H_0 is true and the best explanation for the data if no assumption is made on μ . The denominator is simply the likelihood function evaluated at the usual maximum likelihood estimator. To find the numerator we have to find the maximum likelihood estimator of the background rate b assuming that the signal rate is known to be μ_0 .

$$\frac{\partial}{\partial b} \log l(\mu_0, b; x, y) = \frac{x}{\mu_0 + b} - 1 + \frac{y}{b} - \tau \doteq 0$$

$$\hat{b} = \frac{x + y - (1 + \tau)\mu_0 + \sqrt{(x + y - (1 + \tau)\mu_0)^2 + 4(1 + \tau)y\mu_0}}{2(1 + \tau)}$$

$l(\mu, \hat{b}; x, y)$ is called the profile likelihood function of μ . The usefulness of the likelihood ratio test statistic lies in the fact that approximately we have

$$-2 \log \Lambda(\mu_0; x, y) \sim \chi^2(d)$$

that is $-2 \log \Lambda$ has an approximate Chi-Square distribution with d degrees of freedom, where d is the number of parameters in the model minus the number of parameters specified in the null hypothesis. Here we have $d = 1$. Because the profile likelihood l differs from the likelihood ratio Λ only by a constant independent of μ_0 , we can concentrate on the profile likelihood. For more details on the likelihood ratio test statistic see Casella and Berger [5]. For information on the profile likelihood see Bartlett [6], Lawley [7] and Murphy and Van Der Vaart [8]. In figure 2 we have the profile likelihood function for

the case $x = 6$, $y = 2$, $\tau = 2$. To find a $(1 - \alpha) \cdot 100\%$ confidence interval we start at the minimum, which of course is at the usual maximum likelihood estimator, and then move to the left and to the right until the function rises by the α percentile of a χ^2 distribution with 1 degree of freedom.

The method here uses an approximation to the profile likelihood function by a quadratic function. Unfortunately, in cases where the number of observations in the signal region is small compared to the number of background events, the profile likelihood function becomes almost linear and this approximation does not work. For those cases we will use the following method:

We overlay the confidence region described previously with the profile likelihood curve (μ, \hat{b}) . Then we find the smallest value of μ that is on this curve but not in the confidence region. Figure 3 illustrates this method. Clearly, in the case of fewer observations in the signal region than in the background region only an upper bound will be quoted. We will use this second method whenever the profile likelihood function has a positive derivative at $\mu = 0$. It turns out that the limits obtained by these two methods are compatible. Figure 4 shows the upper bound for a number of cases with the limits obtained by the confidence region method drawn as squares and the limits using the likelihood ratio method as diamonds. The transition from one method to the other is quite smooth.

3 Extensions of the Method

3.1 *Estimating background from Monte Carlo*

Our method can also be applied when a Monte Carlo with limited statistics is used to estimate the background rate. Assume we run the Monte Carlo n times and observe a total of y events. In the data we see x events in the signal region. Then a probability model for this situation is given by:

$$X \sim \text{Pois}(\mu + b), Y \sim \text{Pois}(nb)$$

We notice, of course, that this is actually the same model as the one previously used, only with an n instead of a τ . Therefore, we can use our method for this situation without any changes.

3.2 *Include a second background source*

Sometimes there is a second source of background present in the data, this one characterized by the fact that it only appears in the signal region. An example is an invariant mass histogram where some of the background comes from the misidentification of pions with muons. Say we run a Monte Carlo n times and observe a total of z events of this type. In the data we have x observations in the signal region and y observations in a suitably chosen background region.

The probability model for this case is given by :

$$X \sim Pois(\mu + b + \eta), Y \sim Pois(\tau b)$$

$$Z \sim Pois(n\eta)$$

where μ is the signal rate, b is the rate of the first background source and η is the rate of the second background source. Then:

$$P_{\mu,b,\eta}(X = x, Y = y, Z = z) = \frac{(\mu + b + \eta)^x}{x!} e^{-(\mu+b+\eta)} \frac{(\tau b)^y}{y!} e^{-\tau b} \frac{(n\eta)^z}{z!} e^{-n\eta}$$

We can extend our method to this situation in a fairly straightforward manner. First, we have to find the profile likelihood function, which leads to the following nonlinear system of equations:

$$\frac{\partial \log l}{\partial b} = \frac{x}{\mu + b + \eta} - 1 + \frac{y}{b} - \tau = 0$$

$$\frac{\partial \log l}{\partial \eta} = \frac{x}{\mu + b + \eta} - 1 + \frac{z}{\eta} - n = 0$$

It turns out that $\hat{\eta}$ can be found as the second largest root of the cubic equation

$ax^3 + bx^2 + cx + d = 0$ with

$$a = (1 + n)(n - \tau)$$

$$b = (x + z - (1 + n)\mu)(\tau - n) - (1 + n)(z + y)$$

$$c = xz + (\tau - n)z\mu + z^2 + yz - (1 + n)z\mu$$

$$d = \mu z^2$$

and then \hat{b} is given by

$$\hat{b} = \frac{x + y - (1 + \tau)(\mu + \hat{\eta}) + \sqrt{(x + y - (1 + \tau)(\mu + \hat{\eta}))^2 + 4(1 + \tau)y(\mu + \hat{\eta})}}{2(1 + \tau)}$$

The same problem as before arises again in this situation: in the case of few events in the signal region the profile likelihood function is nearly linear. We can use the same remedy as before by instead finding the intersection of the boundary of the three-dimensional confidence region in (μ, b, η) space with the profile likelihood curve $(\mu, \hat{b}, \hat{\eta})$.

4 Performance of this Method

In this section we will study the true coverage and the power of this method. For comparison we will use the unified approach of Feldman and Cousins [1]. Although that method was not designed to deal with uncertainty in the background, it is the standard method used in calculating confidence intervals in High Energy Physics at this time according to the Review of Particle Physics [9]. The coverage rates shown here were obtained through exact computation, without approximation. In figure 5 we have the case of one background region of equal size to the signal region, and finding 90% confidence intervals. The true background rates are $b = 1, 3$ and 5 and the signal rates go from 0 to 5. Clearly our method has much better coverage than Feldman and Cousins [1]; although, due to the approximation used in the method, the coverage is

sometimes slightly worse than the nominal one. The ragged appearance of the graph is due to the discrete nature of the Poisson distribution and in general is unavoidable. Figure 6 has the case of two background regions and a 99% confidence interval. Again the new method performs very well, whereas Feldman and Cousins [1] does not achieve the nominal coverage.

Next, we illustrate the performance of this method when the background rate is estimated by Monte Carlo. In figure 7 we compute the coverage rates for the cases $\mu = 0$ and $b = 0.5, 1.5$ and 2.5 . The Monte Carlo is run n times where we let n range from 1 to 10. Our method performs satisfactorily for all cases. This plot is also interesting because it gives some insight into the number of runs of the Monte Carlo needed before one can assume that the background rate is known.

The performance of the extension of our method to the case of two different background sources discussed in section 3.2 has to be studied using mini Monte Carlo because the number of possible observations (x, y, z) becomes quite large. We have run a variety of these mini Monte Carlo studies. In Figure 8 we show the results for the case $\tau = 2$, $n = 10$ and a 90% confidence interval. The true coverage rates of our method appear to be in line with the nominal ones, again with a few cases where the coverage is slightly worse due to the approximation used in the method. Due to the discreteness of the Poisson distribution the method is also quite conservative for many situations.

As Feldman and Cousins [1] noted, there has been some criticism of their limits in cases where an experiment finds fewer events than are expected from the background rate. For example, an experiment with an expected background rate of $b = 0.5$ and no observed events would quote a Feldman-Cousins upper limit of 1.94; whereas an experiment with an expected background rate of $b = 1.5$ and no observed events would quote a Feldman-Cousins upper limit of 1.33. Our tables show a similar effect. Feldman and Cousins [1] explain this apparent inconsistency in some detail, and we are in full agreement with their reasoning.

It may also come as somewhat of a surprise that our method yields limits that are sometimes smaller than the limits in Feldman and Cousins [1]. Contrary to the instance discussed in the previous paragraph, this happens in some cases where more events are observed in the signal region than in the sidebands. For example, consider the following situation: say we use one sideband of equal size to the signal region, that is we have $\tau = 1$, and we observe $x = 1$ events in the signal region and $y = 0$ events in the sideband. Then the 90% upper limit using our method is found to be 3.65, whereas assuming $b = 0$ and using Feldman and Cousins method gives a 90% upper limit of 4.36, or about 19% larger. So here we get a smaller limit despite the fact that we have an additional uncertainty. This apparent paradox can be explained as follows: Let us consider the question whether the single event observed in the signal region is a signal or a background event. Using Feldman and Cousins [1] the answer

is clear: having assumed $b = 0$ we know that there is no background, therefore this event has to be from the signal. In fact, it would be quite reasonable in this situation to actually quote a lower limit larger than 0, because we already have proof that $\mu > 0$. On the other hand, in our method $y = 0$ does not imply $b = 0$, it only means that b is not too large. It is therefore still possible that the event in the signal region is in fact a background event, and that $\mu = 0$. It should come as no surprise, then, that in those cases we are quoting a smaller upper limit for μ than Feldman and Cousins [1]. Notice that in theory it would still be preferable to have absolute knowledge of the background rate because one could then make an actual observation of the signal rate rather than having to settle for setting a limit. In practice it is usually impossible to have such precise knowledge and, of course, nobody would claim a discovery based on just one event.

5 Conclusion

The methods for quoting limits for rare decays, mainly Feldman and Cousins [1] and its variants, all suffer from the requirement that the background source be known with a high degree of precision. We have described a new method based on the likelihood ratio test which treats the background uncertainty as a statistical error. The performance of the method is shown to be quite good, with the true coverage rates close to the nominal ones and good power. The

method can be used in situations where the background rate has been estimated from data sidebands as well as where it has been obtained from Monte Carlo. The method can also be extended to cases where a second background source solely appearing in the signal region is present.

In the appendix we provide tables for the confidence intervals and the experimental sensitivity as discussed in Feldman and Cousins [1] and in Review of Particle Physics [9] for the cases $\tau = 1, 2$ and $\alpha = 0.9, 0.99$. The currently used algorithm for computing the limits becomes unreliable in some extreme cases, and in those cases we use NA in the tables. A FORTRAN program for the computation of the limits for any other case as well as for the extension discussed in section 4.2. can be obtained by writing to w_rolke@rumac.upr.clu.edu.

References

- [1] R.D. Cousins, G.J. Feldman, "A Unified Approach to the Classical Statistical Analysis of Small Signals", *Phys. Rev*, **D57**, (1998) 3873.
- [2] C. Giunti, "A new ordering principle for the classical statistical analysis of Poisson processes with background" , *Phys. Rev* **D59**, 053001 (1999).
- [3] B.P. Roe, M.B. Woodroffe, "Improved Probability Method for Estimating Signal in the Presence of Background", *Phys. Rev* **D60** 053009 (1999).
- [4] R.D. Cousins, V.L.Highland, "Incorporating systematic uncertainties into an upper limit", *Nucl. Inst. and Methods A320*, (1992) 331-335.

- [5] G. Casella, R.L. Berger, *Statistical Inference*, Duxburry Press, (1990) 346.
- [6] M.S. Bartlett, "Approximate Confidence Intervals, Part II: More than one Unknown Parameter", *Biometrika* **Vol. 40**, (1953) 306-317.
- [7] D.N. Lawley, "A General Method For Approximating To The Distribution Of The Likelihood Ratio Criteria", *Biometrika* **Vol. 43**, (1956) 295-303.
- [8] S.A. Murphy, A.W. Van Der Vaart, "On Profile Likelihood", *Journal of the American Statistical Association*, **Vol. 95**, (2000), 449-485.
- [9] C. Caso *et al.* (Particle Data Group), *Eur. Phys. J. C* **3**, 1 (1998) 177.

6 Appendix

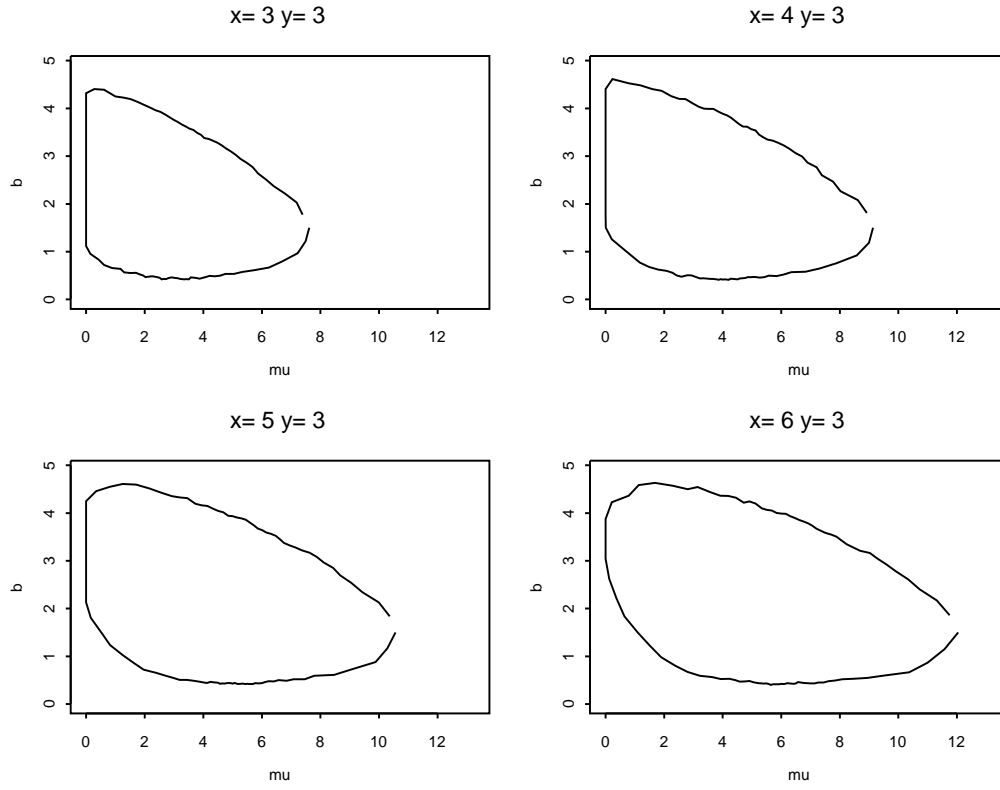


Fig. 1. Two dimensional confidence regions for $x = 3, 4, 5$ and 6 signal events, and $y = 3$ background events. The background region is twice the size of the signal region, and a 90% confidence level was used.

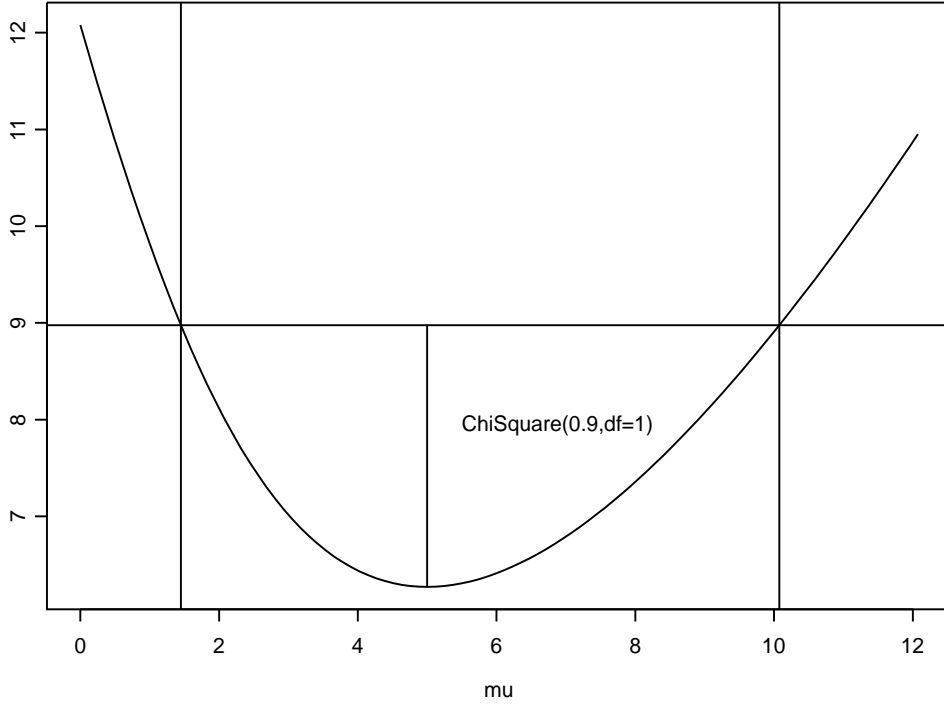


Fig. 2. Profile likelihood function with two-sided limits for the case of $x = 6$ events in the signal region and $y = 2$ events in the background region. The background region is twice the size of the signal region ($\tau = 2$). The nominal coverage probability is 0.9.

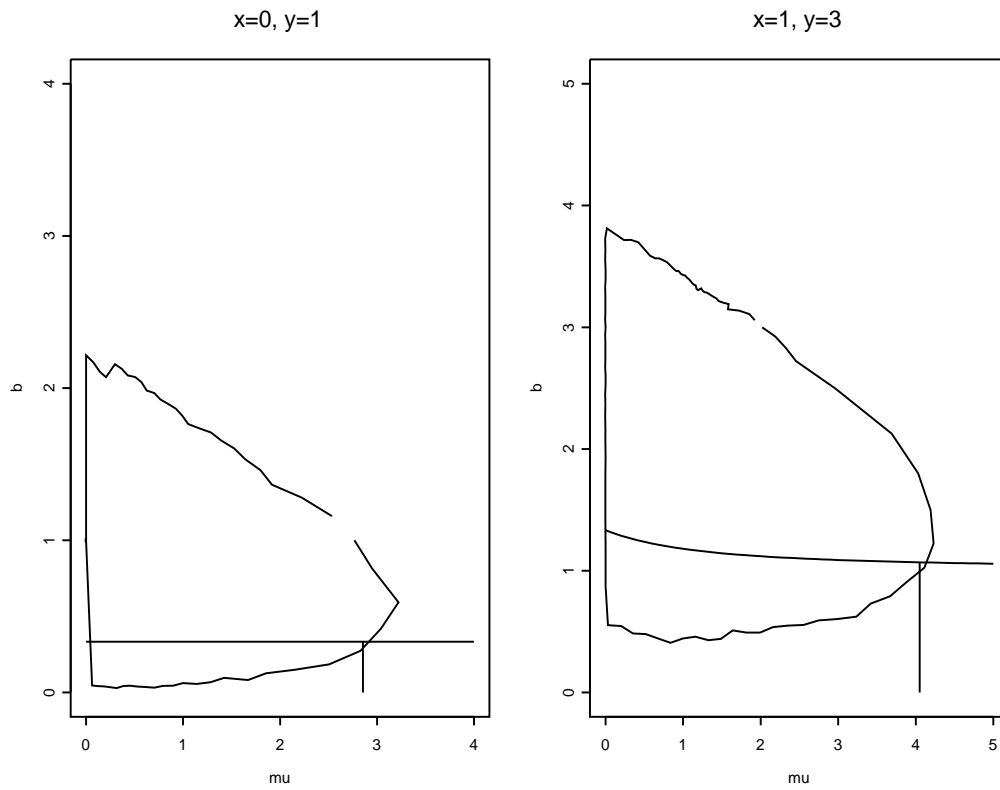


Fig. 3. Confidence region with profile likelihood curve and upper bounds for the cases $x = 0, y = 1$ and $x = 1, y = 3$.

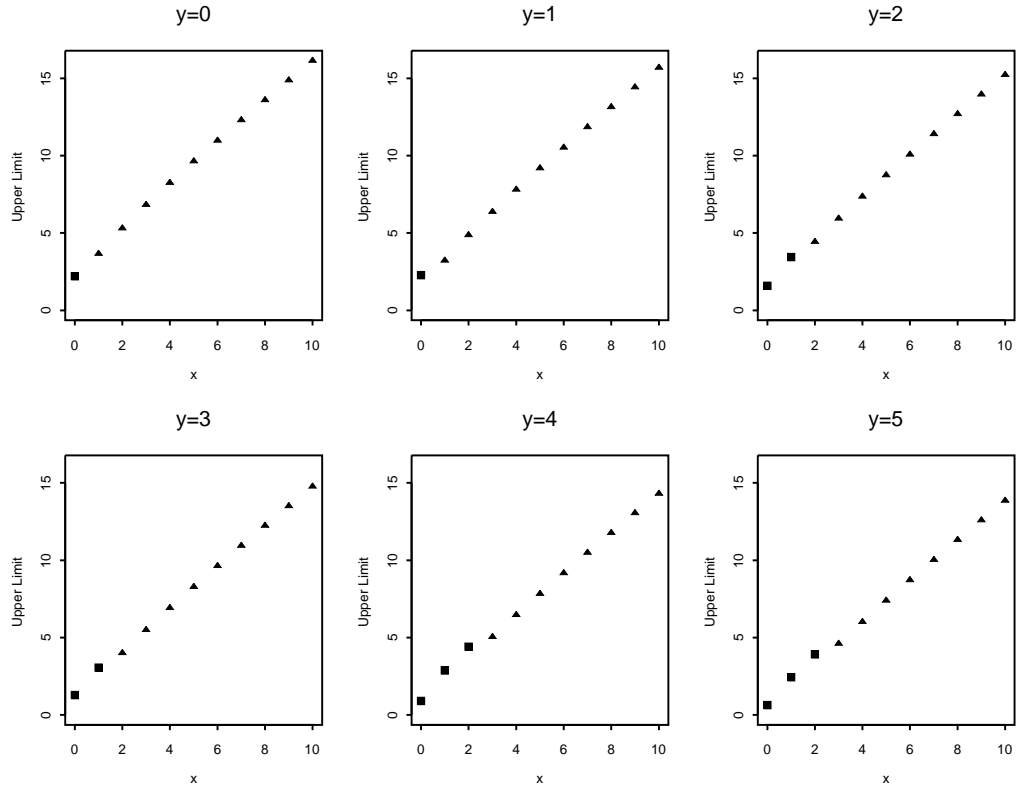


Fig. 4. Upper limits obtained for a range of x and y values. Squares indicate that the confidence region method was used whereas diamonds indicate that the profile likelihood method was used.

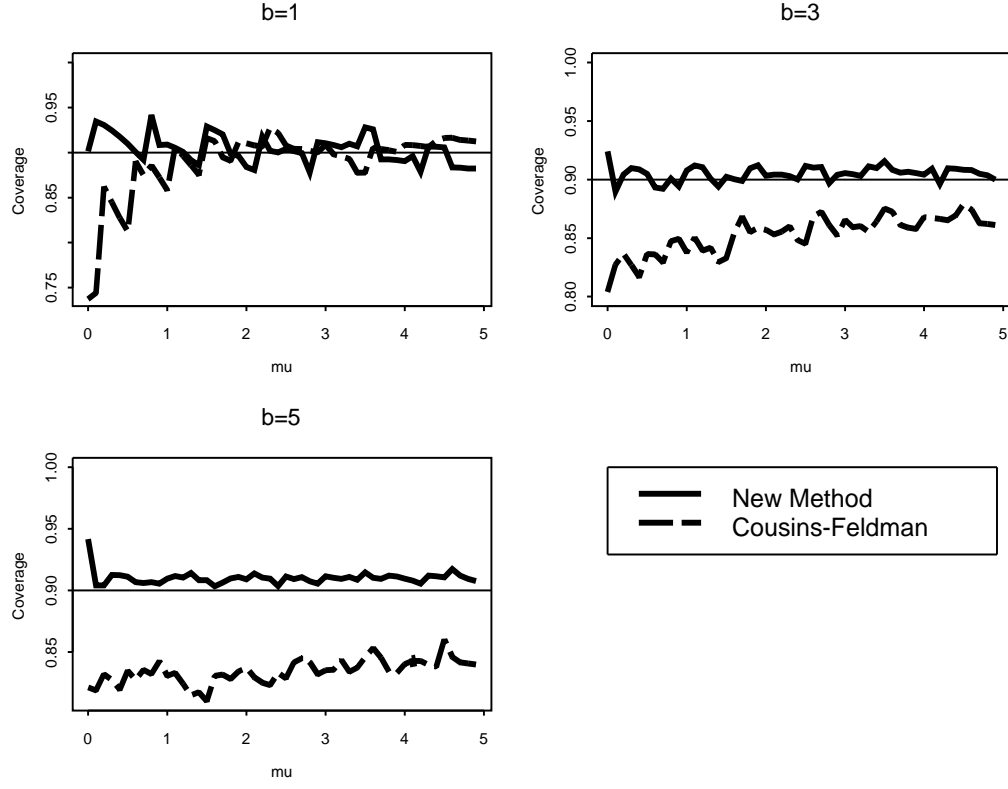


Fig. 5. True coverage rates for our method and for the unified approach of Feldman and Cousins. The background rates are $b = 1, 3$ and 5 . The signal rates range from 0 to 5 . We have $\tau = 1$ and we use 90% confidence intervals.

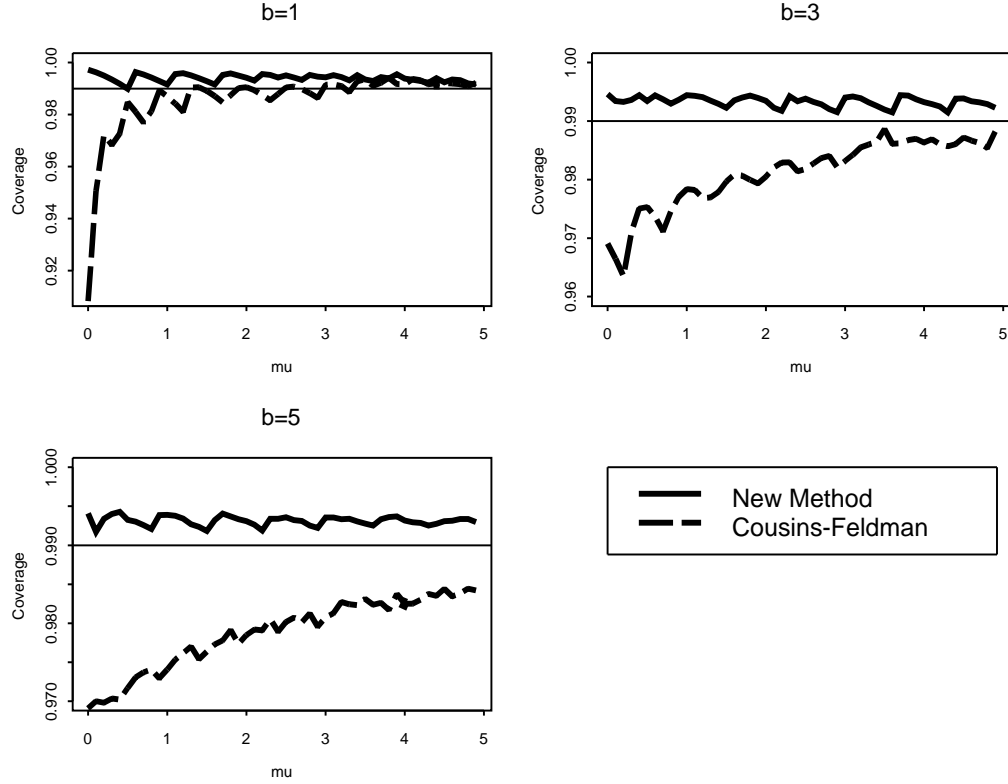


Fig. 6. True coverage rates for our method and for the unified approach of Feldman and Cousins. The background rates are $b = 1, 3$ and 5 . The signal rates range from 0 to 5 . We have $\tau = 2$ and we use 99% confidence intervals.

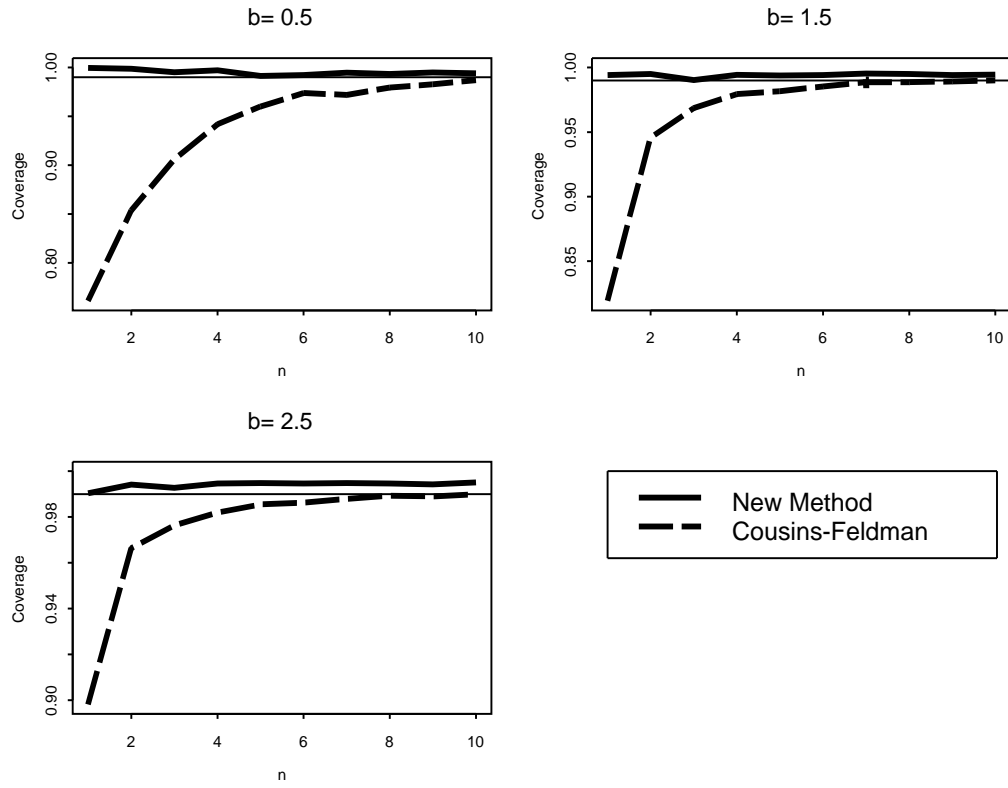


Fig. 7. Here the background is estimated with Monte Carlo. There is no signal present, and the Monte Carlo was run n times. 99% confidence intervals are used.

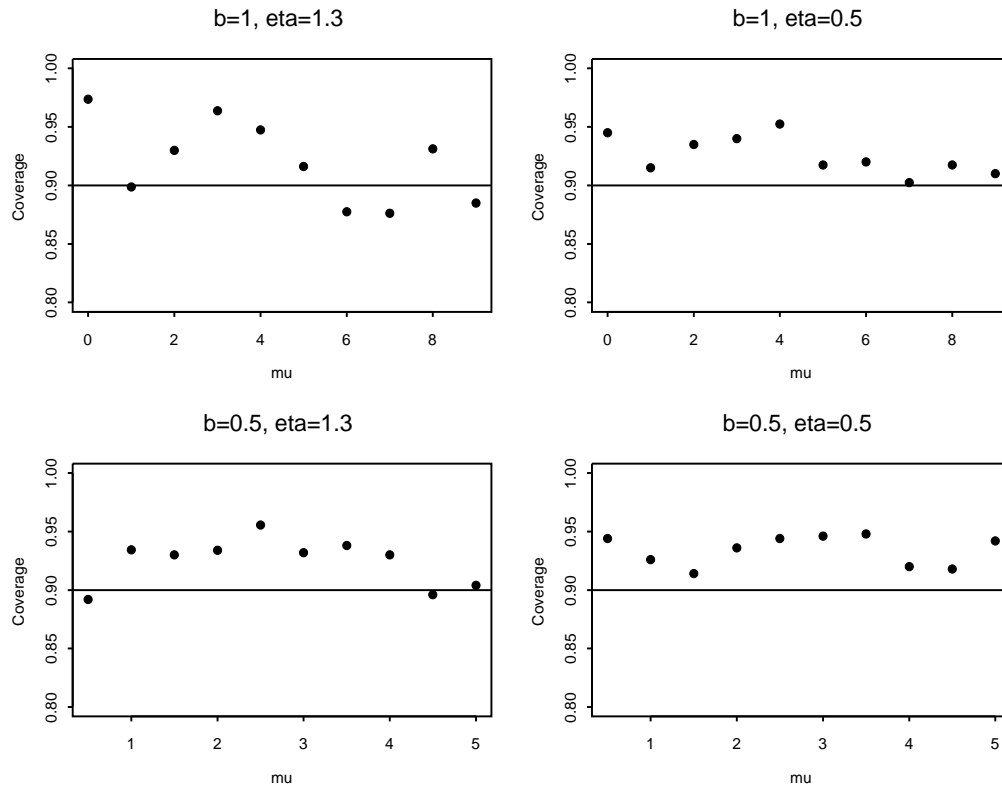


Fig. 8. The case of two different background sources. Coverage rates are based on mini Monte Carlo. The simulation is for the case with a background region twice the size of the signal region. The Monte Carlo was done 10 times and a 90% confidence interval was found.

Table 1: 90% CI for Poisson signal μ with $\tau = 1$. y is the number of events observed in the background region and x is the number of events observed in the signal region. In this case the estimated background rate would be $b = y$.

y					
x	0	1	2	3	4
0	0,2.21	0,1.71	0,0.86	0,NA	0,NA
1	0,3.65	0,3.62	0,2.71	0,1.98	0,1.3
2	0.03,5.3	0,4.52	0,4.19	0,3.41	0,2.74
3	0.73,6.81	0,6.02	0,5.22	0,4.81	0,4.17
4	1.42,8.25	0,7.44	0,6.63	0,5.82	0,5.57
5	2.11,9.63	0.16,8.81	0,7.99	0,7.17	0,6.35
6	2.81,10.98	0.93,10.15	0,9.32	0,8.49	0,7.66
7	3.5,12.3	1.7,11.46	0.22,10.63	0,9.79	0,8.96
8	4.2,13.59	2.49,12.76	1.02,11.92	0,11.07	0,10.23
9	4.92,14.87	3.27,14.03	1.82,13.19	0.45,12.34	0,11.49
10	5.66,16.14	4.06,15.29	2.63,14.44	1.27,13.59	0,12.74
11	6.41,17.39	4.85,16.54	3.44,15.69	2.09,14.83	0.78,13.97
12	7.17,18.63	5.65,17.78	4.26,16.92	2.92,16.06	1.61,15.2
13	7.94,19.86	6.45,19	5.08,18.14	3.75,17.28	2.45,16.42
14	8.71,21.09	7.26,20.23	5.9,19.36	4.58,18.5	3.29,17.63
15	9.5,22.3	8.07,21.44	6.72,20.57	5.41,19.7	4.13,18.83
16	10.29,23.51	8.89,22.64	7.55,21.77	6.25,20.9	4.97,20.03
17	11.09,24.71	9.71,23.84	8.38,22.97	7.09,22.1	5.82,21.22
18	11.89,25.91	10.53,25.03	9.21,24.16	7.93,23.29	6.67,22.41
19	12.7,27.1	11.35,26.22	10.05,25.35	8.77,24.47	7.52,23.59
20	13.52,28.28	12.18,27.41	10.89,26.53	9.62,25.65	8.37,24.77
y					
x	5	6	7	8	9
0	0,NA	0,NA	0,NA	0,NA	0,NA
1	0,0.59	0,NA	0,NA	0,NA	0,NA
2	0,2.12	0,1.36	0,0.41	0,NA	0,NA
3	0,3.52	0,2.66	0,1.9	0,1.05	0,0.16
4	0,4.86	0,4.02	0,3.26	0,2.39	0,1.5
5	0,6.32	0,5.39	0,4.67	0,3.72	0,2.86
6	0,6.83	0,6.8	0,6.02	0,4.97	0,4.17
7	0,8.12	0,7.27	0,7.27	0,6.32	0,5.54
8	0,9.39	0,8.54	0,7.69	0,7.68	0,6.83
9	0,10.64	0,9.79	0,8.94	0,8.08	0,8.05
10	0,11.88	0,11.03	0,10.17	0,9.31	0,8.45
11	0,13.12	0,12.26	0,11.4	0,10.53	0,9.67
12	0.33,14.34	0,13.48	0,12.61	0,11.75	0,10.88
13	1.17,15.55	0,14.69	0,13.82	0,12.95	0,12.08
14	2.02,16.76	0.76,15.89	0,15.02	0,14.15	0,13.28
15	2.86,17.96	1.62,17.09	0,38,16.22	0,15.35	0,14.47
16	3.71,19.16	2.47,18.28	1.24,17.41	0.02,16.53	0,15.66
17	4.57,20.35	3.33,19.47	2.1,18.59	0.88,17.72	0,16.84
18	5.42,21.53	4.18,20.65	2.96,19.77	1.75,18.9	0.54,18.01
19	6.27,22.71	5.04,21.83	3.83,20.95	2.62,20.07	1.41,19.19
20	7.13,23.89	5.91,23.0	4.69,22.12	3.48,21.24	2.28,20.35

Table 2: 90% CI for Poisson signal μ with $\tau = 2$. y is the number of events observed in the background region and x is the number of events observed in the signal region. In this case the estimated background rate would be $b = y/2$.

y					
x	0	1	2	3	4
0	0,2.21	0,2.27	0,1.58	0,1.28	0,0.9
1	0,3.65	0,3.22	0,3.44	0,3.06	0,2.87
2	0.42,5.3	0,4.87	0,4.43	0,4	0,4.41
3	0.97,6.81	0.08,6.37	0,5.93	0,5.49	0,5.05
4	1.54,8.25	0.74,7.8	0.03,7.36	0,6.91	0,6.47
5	2.16,9.63	1.42,9.18	0.74,8.73	0.07,8.28	0,7.83
6	2.82,10.98	2.12,10.53	1.45,10.08	0.8,9.62	0.17,9.17
7	3.5,12.3	2.83,11.84	2.18,11.39	1.54,10.94	0.91,10.48
8	4.2,13.59	3.55,13.14	2.91,12.68	2.29,12.23	1.67,11.77
9	4.92,14.87	4.29,14.42	3.66,13.96	3.04,13.5	2.43,13.05
10	5.66,16.14	5.03,15.68	4.42,15.22	3.81,14.77	3.21,14.31
11	6.41,17.39	5.79,16.93	5.18,16.47	4.58,16.01	3.98,15.56
12	7.17,18.63	6.56,18.17	5.96,17.71	5.36,17.25	4.77,16.79
13	7.94,19.86	7.34,19.4	6.74,18.94	6.15,18.48	5.56,18.02
14	8.71,21.09	8.12,20.63	7.53,20.16	6.94,19.7	6.36,19.24
15	9.5,22.3	8.91,21.84	8.32,21.38	7.74,20.91	7.16,20.45
16	10.29,23.51	9.71,23.05	9.12,22.58	8.54,22.12	7.97,21.65
17	11.09,24.71	10.51,24.25	9.93,23.78	9.35,23.32	8.78,22.85
18	11.89,25.91	11.32,25.44	10.74,24.98	10.17,24.51	9.6,24.05
19	12.7,27.1	12.13,26.63	11.56,26.17	10.99,25.7	10.42,25.23
20	13.52,28.28	12.95,27.82	12.38,27.35	11.81,26.88	11.24,26.42

y					
x	5	6	7	8	9
0	0,0.64	0,0.03	0,NA	0,NA	0,NA
1	0,2.44	0,1.96	0,1.55	0,1.01	0,0.66
2	0,3.92	0,3.51	0,3.02	0,2.68	0,2.26
3	0,4.61	0,5.02	0,4.6	0,4.14	0,3.74
4	0,6.02	0,5.57	0,5.13	0,5.55	0,5.17
5	0,7.39	0,6.94	0,6.49	0,6.04	0,5.59
6	0,8.72	0,8.27	0,7.82	0,7.37	0,6.91
7	0.3,10.03	0,9.58	0,9.12	0,8.67	0,8.21
8	1.06,11.32	0.45,10.86	0,10.41	0,9.95	0,9.5
9	1.83,12.59	1.23,12.13	0.63,11.68	0.04,11.22	0,10.76
10	2.61,13.85	2.01,13.39	1.42,12.93	0.83,12.47	0.25,12.01
11	3.39,15.09	2.8,14.64	2.21,14.18	1.63,13.72	1.04,13.25
12	4.18,16.33	3.59,15.87	3.01,15.41	2.43,14.95	1.85,14.49
13	4.98,17.56	4.39,17.09	3.81,16.63	3.23,16.17	2.66,15.71
14	5.78,18.77	5.2,18.31	4.62,17.85	4.04,17.39	3.47,16.92
15	6.58,19.99	6.01,19.52	5.43,19.06	4.86,18.59	4.29,18.13
16	7.39,21.19	6.82,20.73	6.25,20.26	5.68,19.8	5.11,19.33
17	8.21,22.39	7.64,21.92	7.07,21.46	6.5,20.99	5.93,20.53
18	9.03,23.58	8.46,23.11	7.89,22.65	7.33,22.18	6.76,21.72
19	9.85,24.77	9.28,24.3	8.72,23.83	8.15,23.37	7.59,22.9
20	10.68,25.95	10.11,25.48	9.55,25.02	8.99,24.55	8.43,24.08

Table 3: 99% CI for Poisson signal μ with $\tau = 1$. y is the number of events observed in the background region and x is the number of events observed in the signal region. In this case the estimated background rate would be $b = y$.

y					
x	0	1	2	3	4
0	0,4.52	0,4.41	0,3.6	0,2.75	0,2.05
1	0,6.13	0,6.7	0,5.67	0,5.09	0,4.33
2	0,8.12	0,7.37	0,7.51	0,7	0,6.26
3	0,9.9	0,9.14	0,8.38	0,8.7	0,7.85
4	0,11.56	0,10.79	0,10.02	0,9.25	0,9.48
5	0.15,13.15	0,12.38	0,11.59	0,10.81	0,10.03
6	0.84,14.69	0,13.9	0,13.12	0,12.33	0,11.54
7	1.53,16.18	0,15.39	0,14.6	0,13.8	0,13.01
8	2.23,17.65	0,16.85	0,16.05	0,15.25	0,14.45
9	2.92,19.08	0.46,18.28	0,17.47	0,16.67	0,15.86
10	3.61,20.49	1.2,19.68	0,18.87	0,18.07	0,17.25
11	4.31,21.88	1.94,21.07	0.16,20.26	0,19.44	0,18.63
12	5.0,23.26	2.68,22.44	0.93,21.63	0,20.81	0,19.99
13	5.69,24.62	3.43,23.8	1.69,22.98	0.08,22.16	0,21.33
14	6.39,25.97	4.18,25.14	2.46,24.32	0.86,23.49	0,22.67
15	7.08,27.3	4.92,26.47	3.22,25.65	1.64,24.82	0.13,23.99
16	7.77,28.62	5.68,27.8	3.99,26.97	2.42,26.14	0.92,25.3
17	8.47,29.94	6.43,29.11	4.76,28.27	3.21,27.44	1.71,26.61
18	9.16,31.24	7.19,30.41	5.54,29.58	3.99,28.74	2.51,27.9
19	9.87,32.54	7.94,31.7	6.31,30.87	4.78,30.03	3.3,29.19
20	10.58,33.83	8.71,32.99	7.09,32.15	5.57,31.31	4.1,30.47

y					
x	5	6	7	8	9
0	0,1.33	0,0.76	0,NA	0,NA	0,NA
1	0,3.71	0,2.94	0,2.14	0,1.44	0,0.8
2	0,5.42	0,4.75	0,3.97	0,3.22	0,2.56
3	0,7.11	0,6.35	0,5.62	0,4.95	0,4.14
4	0,8.69	0,7.99	0,7.38	0,6.52	0,5.72
5	0,10.35	0,9.56	0,8.81	0,8.11	0,7.39
6	0,10.75	0,11.17	0,10.31	0,9.59	0,8.95
7	0,12.21	0,11.41	0,11.84	0,11.05	0,10.36
8	0,13.64	0,12.84	0,12.03	0,12.43	0,11.78
9	0,15.05	0,14.25	0,13.43	0,12.62	0,13.2
10	0,16.44	0,15.63	0,14.82	0,14	0,13.18
11	0,17.81	0,17	0,16.18	0,15.36	0,14.54
12	0,19.17	0,18.35	0,17.53	0,16.7	0,15.88
13	0,20.51	0,19.69	0,18.86	0,18.04	0,17.21
14	0,21.84	0,21.01	0,20.19	0,19.36	0,18.53
15	0,23.16	0,22.33	0,21.5	0,20.67	0,19.84
16	0,24.47	0,23.64	0,22.81	0,21.97	0,21.14
17	0.26,25.77	0,24.94	0,24.1	0,23.26	0,22.43
18	1.06,27.07	0,26.23	0,25.39	0,24.55	0,23.71
19	1.86,28.35	0.46,27.51	0,26.67	0,25.83	0,24.99
20	2.67,29.63	1.27,28.79	0,27.94	0,27.1	0,26.25

Table 4: 99% CI for Poisson signal μ with $\tau = 2$. y is the number of events observed in the background region and x is the number of events observed in the signal region. In this case the estimated background rate would be $b = y/2$.

y					
x	0	1	2	3	4
0	0,4.52	0,4.77	0,4.46	0,4	0,3.48
1	0,6.13	0,5.71	0,6.55	0,6.12	0,5.83
2	0,8.12	0,7.7	0,7.27	0,6.85	0,7.66
3	0,9.9	0,9.47	0,9.04	0,8.61	0,8.18
4	0.54,11.56	0,11.13	0,10.7	0,10.26	0,9.83
5	1.09,13.15	0,12.72	0,12.28	0,11.85	0,11.41
6	1.64,14.69	0.53,14.25	0,13.81	0,13.38	0,12.94
7	2.19,16.18	1.15,15.75	0.32,15.3	0,14.86	0,14.42
8	2.74,17.65	1.77,17.2	0.97,16.76	0.21,16.32	0,15.88
9	3.31,19.08	2.41,18.64	1.62,18.19	0.88,17.75	0.16,17.31
10	3.9,20.49	3.05,20.05	2.28,19.6	1.55,19.16	0.84,18.71
11	4.51,21.88	3.7,21.44	2.95,20.99	2.23,20.54	1.53,20.1
12	5.14,23.26	4.36,22.81	3.63,22.36	2.92,21.92	2.23,21.47
13	5.78,24.62	5.03,24.17	4.31,23.72	3.62,23.27	2.93,22.82
14	6.44,25.97	5.7,25.51	5,25.07	4.31,24.61	3.64,24.17
15	7.1,27.3	6.39,26.85	5.7,26.4	5.02,25.95	4.36,25.5
16	7.78,28.62	7.08,28.17	6.4,27.72	5.73,27.27	5.07,26.82
17	8.47,29.94	7.78,29.49	7.11,29.03	6.45,28.58	5.8,28.13
18	9.16,31.24	8.49,30.79	7.83,30.34	7.17,29.88	6.53,29.43
19	9.87,32.54	9.2,32.08	8.55,31.63	7.9,31.18	7.26,30.72
20	10.58,33.83	9.92,33.37	9.27,32.92	8.63,32.46	8,32.01

y					
x	5	6	7	8	9
0	0,3.25	0,2.75	0,2.32	0,2.02	0,1.62
1	0,5.32	0,4.91	0,4.58	0,4.26	0,3.83
2	0,7.31	0,6.83	0,6.54	0,6.15	0,5.63
3	0,7.75	0,8.68	0,8.24	0,7.92	0,7.37
4	0,9.4	0,8.97	0,8.53	0,9.54	0,8.95
5	0,10.97	0,10.54	0,10.1	0,9.67	0,9.23
6	0,12.5	0,12.06	0,11.62	0,11.18	0,10.74
7	0,13.98	0,13.54	0,13.1	0,12.66	0,12.22
8	0,15.43	0,14.99	0,14.55	0,14.11	0,13.66
9	0,16.86	0,16.42	0,15.97	0,15.53	0,15.08
10	0.15,18.26	0,17.82	0,17.37	0,16.93	0,16.48
11	0.85,19.65	0.17,19.2	0,18.75	0,18.31	0,17.86
12	1.55,21.02	0.89,20.57	0.23,20.12	0,19.67	0,19.23
13	2.26,22.37	1.6,21.92	0.95,21.47	0.3,21.02	0,20.58
14	2.98,23.71	2.32,23.26	1.67,22.81	1.03,22.36	0.39,21.91
15	3.7,25.04	3.05,24.59	2,24.14	1.77,23.69	1.13,23.24
16	4.42,26.36	3.78,25.91	3.14,25.46	2.51,25.01	1.87,24.55
17	5.15,27.67	4.51,27.22	3.88,26.77	3.25,26.31	2.62,25.86
18	5.89,28.98	5.25,28.52	4.62,28.07	3.99,27.61	3.37,27.16
19	6.62,30.27	5.99,29.81	5.37,29.36	4.75,28.9	4.13,28.45
20	7.37,31.55	6.74,31.1	6.12,30.64	5.5,30.19	4.88,29.73

Table 5: Experimental sensitivity (defined as the average upper limit that would be obtained by an ensemble of experiments with the observed number of events y in the sidebands and no true signal). Note that the estimated background rate would be $b = y/\tau$.

y	90% C.L.		99% C.L.	
	$\tau = 1$	$\tau = 2$	$\tau = 1$	$\tau = 2$
0	2.21	2.21	4.52	4.52
1	3.31	2.82	6.22	5.35
2	3.97	3.17	7.12	6.14
3	4.62	3.49	8.17	6.45
4	5.29	4.03	8.99	7.12
5	5.90	4.23	9.77	7.43
6	6.37	4.60	10.52	8.00
7	6.86	4.79	11.21	8.30
8	7.23	5.14	11.85	8.86
9	7.64	5.32	12.51	9.02
10	8.01	5.62	13.04	9.52
11	8.40	5.78	13.55	9.76
12	8.71	6.08	14.15	10.14
13	9.09	6.19	14.63	10.37
14	9.39	6.45	15.14	10.76
15	9.71	6.59	15.64	10.96